III Valuation spaces

We have seen how to classify special classes of noninvertible germs of $\mathbb{C}^2$, but what can be said in general?

Idea: instead of holomorphic conjugacies, consider ~~birational~~ bimeromorphic conjugacies:

$$\begin{array}{ccc} X_\pi & \overset{f_\pi}{\dashrightarrow} & X_\pi \\ \downarrow \pi & & \downarrow \pi \\ (\mathbb{C}^2,0) & \xrightarrow{f} & (\mathbb{C}^2,0) \end{array}$$

look for π proper bimeromorphic map, isomorphism outside 0, (called modification) so that the lift f_π "has good properties"

A modification is called smooth if X_π is a smooth surface. (can call it (good) resolution, w.r.t. definitions in the singular setting)

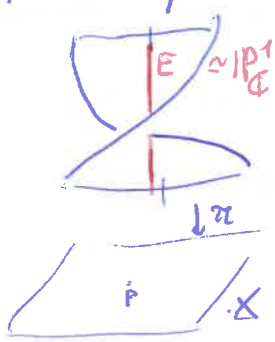
Smooth modifications are compositions of point blow-ups.

Blow-up: it is a local procedure. Given a surface$/\mathbb{C}$ X and a point $p \in X$, the blow-up of X at p is:

- as a set: $\hat{X} = (X \setminus \{p\}) \sqcup \underbrace{\mathbb{P}(T_pX)}_{\overset{\shortparallel}{E} \text{ exceptional divisor}}$, together with a natural

projection $\pi : \hat{X} \to X$ so that $\pi|_{X \setminus \{p\}}$ is the natural inclusion and $\pi(E) = p$.

$\hat{X}$ admits a structure of smooth surface $/\mathbb{C}$, and π is a proper holomorphic map.



(resolution) A smooth modification is a composition of blow-ups of points p_n so that $p_0 = 0$ and p_{j+1} belongs to the exceptional divisor of the ~~...~~ composition of the previous blow-ups.

In local coordinates, $\hat{X}$ can be covered by two charts, so that the projections π are:

$$\pi(x,y) = (x, xy) \quad (E = \{x=0\})$$
$$\pi(x,y) = (xy, y) \quad (E = \{y=0\})$$

Unless differently specified, all modifications are assumed to be smooth.

Given a curve $C \subset (\mathbb{C}^2,0)$, its strict transform C_π is given by $C_\pi = \overline{C \setminus \{0\}}$.

Rem: if C is irreducible, then $C_\pi \cap \pi^{-1}(0)$ consists of a single point.

Modifications in geometry and dynamics.

- Resolution of singularities of varieties (Zariski dim 2, Hironaka higher dimensions)
- Reduction of singularities of foliations:
  - vector fields / 1-forms dim 2 : Seidenberg (~1960) (~ Camacho-Sad)
  - 1-forms in dim 3 : Cano (~1980-90)
  - vector fields in dim 3 : McQuillan - Panazzolo (~2000)
- Reduction of singularities of tangent to the identity germs in $\mathbb{C}^2$. (2000-2010)
- study of non-invertible germs in $\mathbb{C}^2$: Favre-Jonsson. (Gignac-R.)

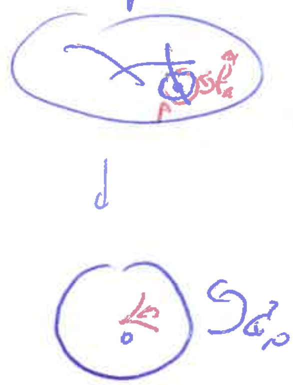
Rigidification.

Thm (Favre-Jonsson, 2007): let $f:(\mathbb{C}^2,0)\circlearrowleft$ be a superattracting germ. Then $\exists\, \pi: X_\pi \to (\mathbb{C}^2,0)$, $\exists\, p \in \pi^{-1}(0)$ s.t. the lift $f_\pi = \pi^{-1}\circ f \circ \pi : X_\pi \circlearrowleft$ defines a rigid germ at p.

Rem: in general f_π is not holomorphic, has indeterminacy points ($\mathrm{Ind}(f_\pi)$). There are maps f so that $\forall \pi \neq \mathrm{id}$, ~~$f_\pi$ has~~ $\mathrm{Ind}(f_\pi) \neq \emptyset$.

Rem (R-) the theorem holds also for semi-superattracting germs, in this case formal normal forms can be given.

Rem: $(f_\pi)_p$ is not necessarily contracting, there are no normal forms in this case.



Algebraic Stability

Let $\pi : X_\pi \to (\mathbb{C}^2, 0)$ be a (smooth) modification.

As we said, in general $\mathrm{Ind}(f_\pi) \neq \emptyset$. This prevents the pull back of exceptional divisors to be functorial:

Denote by $\mathcal{E}(\pi)$ the ~~set~~ (vector space) of ~~R~~ exceptional $\mathbb{R}$-divisors of π:

If $\pi^{-1}(0) = \{E_1, \ldots, E_r\}$ – then $\mathcal{E}(\pi) = \bigoplus_{i=1}^{r} \mathbb{R} E_i$

irreducible components

Assume that f is finite:

Def: $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is finite if $\nexists\, C \subset (\mathbb{C}^2, 0)$ curve so that $f(C) = 0$

Ex: $f(x,y) = (y - x^3; xy^2)$ is finite; $g(x,y) = (x(y-x^3), xy^2)$ is not finite.

In this case, f induces a pull back action $f^* : \mathcal{E}(\pi) \to \mathcal{E}(\pi)$, as follows.

$$\begin{tikzcd}
 & X_{\pi'} \arrow[dl, "\eta"'] \arrow[dr, "\hat{f}"] \arrow[ddl, bend left, "\pi'"] & \\
X_\pi \arrow[rr, dashed, "f_\pi"] \arrow[d, "\pi"'] & & X_\pi \arrow[d, "\pi"] \\
(\mathbb{C}^2, 0) \arrow[rr, "f"] & & (\mathbb{C}^2, 0)
\end{tikzcd}$$

Fix a modification π. f_π is not proper: by blowing up further the source space (along the indeterminacy points), we ~~may~~ find η so that the lift $\hat{f} : X_{\pi'} \to X_\pi$, where $\pi' = \pi \circ \eta$, is regular (holomorphic).

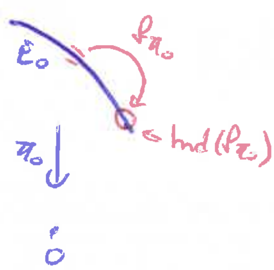
Then we define, $\forall E \in \mathcal{E}(\pi)$, $f^*(E) = \eta_* \hat{f}^*(E)$.

One can check that the definition does not depend on the choice of η.

In general, pull back does not commute with iteration! $(f^n)^* \neq (f^*)^n$.

Example: $f(x,y) = (y - x^3; xy^2)$

We blow-up the origin $\pi_0 : X_{\pi_0} \to (\mathbb{C}^2, 0)$, $E_0 = \pi_0^{-1}(0)$.



Compute f_{π_0} w.r.t. charts (x, xy) and (x, xy):

$$f_{\pi_0}(x,y) = \left(x(y - x^2), \frac{x^2 y^2}{y - x^2}\right)$$ ⇝ Indeterminacy point at $(0,0)$

$f_{\pi_0}(0, y) = (0, 0)$

$\ast$
0

In this case $\mathcal{E}(\pi_0) = \mathbb{R} E_0$, and $f^*(E_0) = E_0$.

In fact, in general: $f^*(E_0) = c E_0$, where

$c = c(f) := \min\left(\mathrm{ord}_0(x \circ f), \mathrm{ord}_0(y \circ f)\right)$ (called attraction rate)

In the example, $c(f) = 1$, and $c(f^2) = 3$

$f^2(x,y) = (xy^2 - (y-x^3)^3, (y-x^3)x^2y^4)$

In particular $(f^2)^* E_0 = 3 E_0 \neq E_0 = (f^*)^{\circ 2}(E_0)$.

In this example: $(f^n)^* = (f^*)^n \ \forall n \iff c(f^n) = c(f)^n$.

Rem: if we allow deformations, this condition is satisfied: it suffices to post compose with a linear map so to avoid $\mathrm{Ind}(f_\pi)$

Def (1): let $f : (\mathbb{C}^2, 0) \circlearrowleft$ be finite and π be a modification.
π is called an algebraically stable model (AS) for f if: $\exists N \gg 0$ s.t. $\forall n \geq N$, $(f^n)^* = (f^N)^* (f^*)^{n-N}$ on $\mathcal{E}(\pi)$.

Algebraic stability can be detected also looking at forward action of f on divisors.

Denote by Γ_π^* the irreducible components of $\pi^{-1}(0)$

Def (2): let $f : (\mathbb{C}^2, 0) \circlearrowleft$ be any (possibly non finite) germ, and π a modification.
π is called an algebraically stable (AS) model for f if $\exists N \gg 0$ s.t.:
$\forall E \in \Gamma_\pi^*$, $\forall n \geq N$, then $f_\pi^n(E) \notin \mathrm{Ind}(f_\pi)$ i.e. $f_\pi^n(E) = p_n \notin \mathrm{Ind}(f_\pi)$ or $f_\pi^n(E) = E_n \in \Gamma_\pi^*$

proper definition through valuations

scheme-theoretically

In the example: $f_\pi^n(E_0) = p$ $\forall n \geq 1$, so π is not AS.
the ind. point.

How to define $f_\pi^*(p)$ when $p \in \mathrm{Ind}(f_\pi)$? as $\bigcap_{\varepsilon > 0} \overline{f_\pi(B_\varepsilon \setminus \{p\})}$.

and $f_\pi(E)$ when $E \cap \mathrm{Ind}(f_\pi) \neq \emptyset$? as $\overline{f_\pi(E \setminus \mathrm{Ind}(f_\pi))}$.

Rem: π is A.S (Def 2) $\Rightarrow$ π is A.S (Def 1).
If f is finite. $\not\Leftarrow$ in general

Moreover, Def 2 makes sense also if f is not finite. We take Def 2 as our definition of A.S models.

* Historical Remarks:

In a global setting, the A.S concept has been introduced by Fornaess-Sibony (1993), as a condition to control cohomology, which allows the construction of invariant objects (measures, currents).

The existence of A.S models has been studied in the following cases:

- Birational maps of $\mathbb{P}^2$ ($\exists$ A.S), one of the ingredients for the study of the degree growth $\deg(f^n)$ for $f \in Cr(\mathbb{P}^2)$. (Cremona group). (Diller-Favre), (see also: Cantat, Blanc, Déserti, Xie)
- Polynomial endomorphisms of $\mathbb{C}^2$ ($\exists$ A.S): Favre-Jonsson (2011) (the local situation uses similar techniques)
- Favre, Hasselblatt-Propp (2007): examples of $\nexists$ A.S models for (2002)
  - monomial rational maps of $\mathbb{P}^2$
  - monomial birational maps of $\mathbb{P}^3$.

* Theorem: (Gignac-R 2013 -2017): $f:(\mathbb{C}^2,0)\circlearrowleft$ non invertible germ (possibly singular)

Then $\forall \pi$ modification, $\exists \pi'$ modification dominating π which is A.S for f.

$$\begin{array}{c} X_{\pi'} \circlearrowleft \text{A.S} \\ \eta \downarrow \\ X_\pi \\ \downarrow \pi \\ (\mathbb{C}^2,0) \end{array}$$

Rem: This can be used together with F-J rigidification result

Rem: $X_{\pi'}$ could be singular (cyclic quotient singularities) it may be assumed smooth up to replacing f by f^2.

Later results: • Jonsson-Wulcan: study of existence of A.S models for toric maps in toric varieties

• Gignac-R (2019): study for $f:(X,x_0)\circlearrowleft$ (X,x_0) normal surface singularity.

We noticed that for π_0 the blow-up of the origin, we have: $f^*(E_0)=c(f)\cdot E_0$.

Since $E_0\cdot E_0=-1$, we can also define $c(f)$ as $-E_0\cdot f^*(E_0)$

By projection formula, we can read this formula in any model, getting $c(f)=-\pi^*E_0\cdot f^*(\pi^*E_0)$.

The theorem on the existence of A.S models, together with Cayley-Hamilton, gives: Corollary (G-R): Let $f:(\mathbb{C}^2,0)\circlearrowleft$ be a non-invertible germ. Then the sequence of attraction rates $c_n=c(f^n)$ eventually satisfies a linear integral recursion relation: $\exists N\gg 0$, $a\in\mathbb{N}, b\in\mathbb{Z}, m\in\mathbb{N}^*$ s.t.

$$c_{n+2m}=a\,c_{n+m}+b\,c_n \quad \forall n\geq N.$$

Rem: $c(f)$ is called attraction rate because:

$$c(f)=\max\{c>0 \mid \|f(p)\|\leq k\cdot\|p\|^c \quad p\to 0,\ k \text{ const}\}$$

An asymptotic version is given by:

Def: The first dynamical degree of f is: $c_\infty(f)=\lambda_1(f)=\lim_{n\to\infty}\sqrt[n]{c(f^n)}$.

Rem: such limit exists since $c(f^{n+m})\geq c(f^n)\cdot c(f^m)$.

Corollary (F-J, 2007): $f:(\mathbb{C}^2,0)\circlearrowleft$, $c(f)$ is a quadratic integer

Rem: true also in the singular case (Gignac-R).

Rem: It is not known if $c(f)$ is an algebraic integer in higher dimensions.

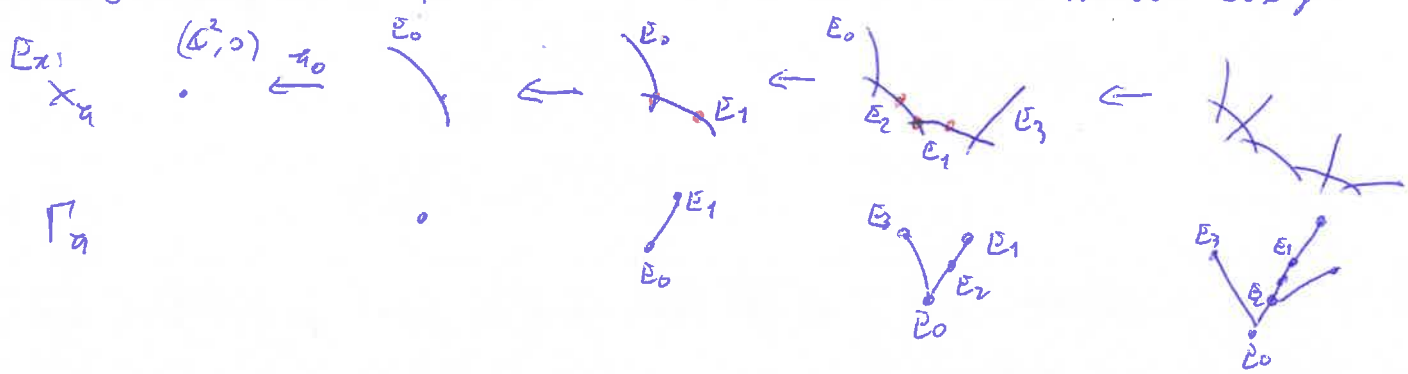
Strategy for the proof of such theorems:

- Define a space $\mathcal{V}$ (the valuative tree) that encodes all possible models (and exceptional primes).
- Define an action $f_\bullet : \mathcal{V} \circlearrowleft$ induced by f that encodes all possible $f_\pi : X_\pi \dashrightarrow X_\pi$.
- Prove some dynamical properties of $f_\bullet$
- Use such properties to prove the theorem.

Dual graphs and the valuative tree.

(Studied in details by Favre-Jonsson, similar objects ~~studied~~ introduced by Contat (Morin-Hamburg spaces), relations with Riemann-Zariski space, Berkovich spaces)

Let π be a modification: To it we can attach a graph Γ_π called dual graph (of π), so that the vertices Γ_π^* are the irreducible components of $\pi^{-1}(0)$ (called exceptional primes), and edges $E — F$, $E \neq F \in \Gamma_\pi^*$ for any intersection point in $E \cap F$ (0 or 1 in this smooth case)



We identify any exceptional prime $E \in \Gamma_\pi^*$ with its strict transform $E_\eta = \eta^* E \in \Gamma_{\pi'}^*$, when $\pi' = \pi \circ \eta$ dominates π.

Dual graphs Γ_π can be embedded into $\mathcal{E}(\pi)$ as follows:

$\forall E \in \Gamma_\pi^v$, we associate the divisor $\dfrac{\check{E}}{b_E}$, where:

- $\check{E}$ is the unique divisor in $\mathcal{E}(\pi)$ such that: $\begin{cases} \check{E}\cdot E = 1 \\ \check{E}\cdot D = 0 \ \ \forall D \in \Gamma_\pi^v,\ D \neq E. \end{cases}$

Rem: $\check{E}$ is well defined because the intersection form on $\mathcal{E}(\pi)$ is negative definite.

- b_E is a normalisation coefficient defined as follows:

  . If $\pi = \pi_0$ is the blow-up of the origin, then $b_{E_0} = 1$.

  . If $\pi > \pi_0$, then write ~~E_0~~ $\pi = \pi_0 \circ \eta$, and $\eta^* E_0 = \sum_{E \in \Gamma_\pi^v} b_E E$.

b_E does not depend on the model where E appears (up to strict transforms)

In general: $\pi^*\mathfrak{m} = \mathcal{O}(-Z_\pi(\mathfrak{m}))$, $Z_\pi(\mathfrak{m}) = \sum_{E \in \Gamma_\pi^v} b_E E$

π log-res. of $\mathfrak{m}$.

For any edge $E - F$, we take the segment $\left[\dfrac{\check{E}}{b_E}, \dfrac{\check{F}}{b_F}\right] \subset \mathcal{E}(\pi)$

This defines an "embedding" $i_\pi : \Gamma_\pi \longrightarrow \Gamma(\pi) \subset \mathcal{E}(\pi)$.

(can be shown by induction)

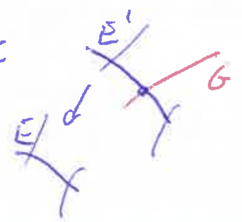
These embeddings are compatible with the identification through strict transform, since if $E' = \eta^* E$, then $\check{E}' = \eta^* \check{E}$

comes from the projection formula

$$(\eta^*\check{E})\cdot D' = \check{E}\cdot \eta_* D' = \begin{cases} 0 & D' = \eta^{-1}(p) \\ 0 & \eta(D') = D \neq E \\ 1 & \eta(D') = E \Rightarrow D' = E_\eta \end{cases}$$

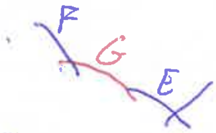
If π is a modification, and η is the blow-up of a point $p \in \pi^{-1}(0)$, we have two cases: (Call G the new exceptional prime ~~div~~ $G = \eta^{-1}(p)$).

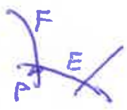
. $\exists!\, E \in \Gamma_\pi^v$, $p \in E$ (free point). In this case: $\check{G} = \check{E}' - G$



$$(\check{E}' - G)\cdot \begin{cases} D \\ E' \\ G \end{cases} = \begin{matrix} 0 - 0 = 0 \\ 1 - 1 = 0 \\ 0 - (-1) = 1 \end{matrix} \qquad \text{here } b_G = b_E$$

- $p \in E \cap F$ (satellite point). In this case $\check{G} = \check{E}' + \check{F}' - G$





$$(\check{E}' + \check{F}' - G)\cdot \begin{cases} D \\ E' \\ F' \\ G \end{cases} = \begin{array}{l} 0+0-0 \\ 1+0-1 \\ 0+1-1 \\ 0+0-(-1) \end{array}$$

here $b_G = b_E + b_F$

It follows that we can define a natural (linear) projection $\underset{\vee}{r_\eta} \; r_{\pi\pi'} : \mathcal{E}(\pi') \to \mathcal{E}(\pi)$, defined as $r_\eta(\check{\eta^* E}) = \check{E}$ $\forall E \in \Gamma_\pi^*$, $r_\eta(G) = 0$, and extended by linearity.

↖ this corresponds to: $r_\eta = \eta_*$.

This r_η sends $\Gamma(\pi')$ surjectively onto $\Gamma(\pi)$.

Hence, we can consider its projective limit $\Gamma := \varprojlim_{\pi} \Gamma(\pi) \hookrightarrow \mathcal{E} := \varprojlim_{\pi} \mathcal{E}(\pi)$

An element of $\mathcal{E}$ is a family $(Z_\pi)_\pi$ of exceptional divisors, satisfying:

$\forall \pi' \geq \pi$ $(\pi' = \pi \circ \eta)$, $\eta_* Z_{\pi'} = Z_\pi$.

Such a family $Z = (Z_\pi)_\pi$ is called a b-divisor.

(It is called Cartier if $\exists \tilde{\pi}$ s.t. $\forall \pi \geq \tilde{\pi}$, $Z_\pi = \eta^* Z_{\tilde{\pi}}$)
(otherwise, Weil) $\qquad \pi = \tilde{\pi} \circ \eta$

The valuative tree can be thought as this subset Γ lying inside $\mathcal{E}$.
More properly, this is the embedding of the valuative tree in the vector space of b-divisors.

Valuations have a more algebraic definition.

Def: Set: $R = \mathbb{C}[[x,y]]$, $\mathfrak{m} = \langle x, y \rangle$ its maximal ideal.

A valuation (more properly, semi-valuation of rank 1) is a map:

$\nu : R \to [0, +\infty]$ satisfying:

- $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ $\quad \forall \phi, \psi \in R$
- $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ $\quad \forall \phi, \psi \in R$

· $\nu(0) = +\infty$, $\nu(\mathbb{C}^*) = 0$.

If $\mathfrak{a}$ is an ideal of R, we set $\nu(\mathfrak{a}) := \inf\{\nu(\phi) \mid \phi \in \mathfrak{a}\}$.

We say that ν is:

· centered, if $\nu(\mathfrak{m}) > 0$. (its set is denoted $\hat{\mathcal{V}}$)

· ~~...~~ finite if moreover $\nu(\mathfrak{m}) < +\infty$ (" " " $\hat{\mathcal{V}}^*$)

· normalised if $\nu(\mathfrak{m}) = 1$ (" " " $\mathcal{V}$).

$\mathcal{V}$ is called the valuative tree.

Rem: The only non-finite centred valuation is the trivial valuation triv_0, defined by: $\nu(\phi) = \begin{cases} +\infty & \phi \in \mathfrak{m} \\ 0 & \phi \notin \mathfrak{m}. \end{cases}$

$\hat{\mathcal{V}}$ is a cone over $\mathcal{V}$ with apex triv_0 (since $\nu \in \hat{\mathcal{V}} \Rightarrow \lambda\nu \in \hat{\mathcal{V}}$ $\forall \lambda \in \mathbb{R}^+$)

$\hat{\mathcal{V}}$ is endowed with a partial order: $\nu \leq \mu \overset{\text{def}}{\Longleftrightarrow} \nu(\phi) \leq \mu(\phi)$ $\forall \phi \in R$.

With this order, $\mathcal{V}$ is a (complete) $\mathbb{R}$-tree. (see Favre-Jonsson book)

$\mathcal{V}$ is also endowed with a topology (called weak), the coarsest so that
$\mathcal{V} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$
$\nu \mapsto \nu(\phi)$ is continuous $\forall \phi \in R$.

Examples:

· ord_0 the multiplicity at 0 is a normalised valuation.

more generally, consider a modification π, and an exceptional prime $E \in \Gamma_\pi^*$.

Then $\nu_E(\phi) := \dfrac{\mathrm{ord}_E(\phi \circ \pi)}{b_E}$ defines a normalised valuation, called divisorial

The easiest example is given by $\mathrm{ord}_0 = \nu_{E_0}$ (E_0 the exceptional divisor of the blow-up of the origin). Divisorial valuations are dense in $\mathcal{V}$.

· More generally, consider π a modification, and $p \in E \cap F$ a satellite point.
Pick local coordinates (z,w) at p so that $E = \{z=0\}$ and $F = \{w=0\}$. For any $r,s>0$ we may consider the monomial valuation at p of weights (r,s), defined on $\mathbb{C}[[z,w]]$ by:

$$\mu^p_{r,s}\left(\sum \varphi_{ij} z^i w^j\right) = \min\{ri + sj \mid \varphi_{ij} \neq 0\}.$$

called quasimonomial $\in \mathcal{V}^*$

We can push down this valuation, and get $\nu^p_{r,s}(\phi) = \mu^p_{r,s}(\phi \circ \pi)$.
This valuation is normalised as far as $r b_E + s b_F = 1$.
If $\frac{s}{r} \in \mathbb{Q}$, $\nu^p_{r,s}$ is actually divisorial. (for $\pi' > \pi$ obtained by further (toric) blow-ups of p.
If $\frac{s}{r} \in \mathbb{R}\setminus\mathbb{Q}$, $\nu^p_{r,s}$ is called irrational.

To these valuations, we can naturally associate a b-divisor $Z(\nu)$.
For divisorial valuations, $Z(\nu_E)$ is the Cartier b-divisor ~~associated~~ determined by $\frac{\check{E}}{b_E}$ on $\mathcal{E}(\pi)$, i.e., the element in $\Gamma(\pi)$ associated to E.

For quasimonomial valuations $Z_\pi(\nu^p_{r,s})$ will be given by $r\check{E} + s\check{F}$.
Notice that if $\nu^p_{r,s}$ is normalised, $r = \frac{t}{b_E}$, $s = \frac{1-t}{b_F}$, and $r\check{E} + s\check{F}$ belongs to the segment $\left[\frac{\check{E}}{b_E}, \frac{\check{F}}{b_F}\right]$.

Taking the projective limit, we get two other kind of valuations

Curve valuations: If $C \subset (\mathbb{C}^2, 0)$ is an irreducible curve, we can define $\nu_C(\phi) = \dfrac{C \cdot \{\phi = 0\}}{m(C)}$ ← intersection multiplicity at 0.

$\nu_C(\phi)$: $\in \mathcal{V}$

$m(C)$: ← multiplicity of C, a normalisation constant.

Its b-divisors can be computed as follows. ← Only valuations s.t. $\{\nu = +\infty\} \supsetneq \{0\}$.

Assume π is an embedded resolution of $(C, 0) \subset (\mathbb{C}^2, 0)$.

This means that $\pi^{-1}(C) = C_\pi \cup \pi^{-1}(0)$ has s.n.c

C_π: ↑ strict transform

In particular, $C_\pi \cap \pi^{-1}(0)$ is a single free point: $\exists! \, E \in \Gamma_\pi^*$, $E \cap C_\pi \neq \emptyset$.

Then $Z_\pi(\nu_C) = \dfrac{\check{E}}{b_E}$ (notice: $b_E = m(C)$)

- Infinitely singular valuations: can be thought as curve valuations of infinite multiplicity, they are maximal elements in $\mathcal{V}$.

Action of f on $\mathcal{V}$

The algebraic setting allows an easy definition of the action induced by $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ on $\mathcal{V}$.

Set: $f_* \nu(\phi) = \nu(f^* \phi) = \nu(\phi \circ f)$

If $\nu \in \mathcal{V}$, then $f_* \nu$ is a centred valuation, but not necessarily finite or normalised.

To have $f_* \nu$ non finite, we need: $\nu(f^* \mathfrak{m}) = +\infty$.

In particular (since $f \neq 0$), ν must be a curve valuation ν_C, and $f(C) = 0$. This will be called a "contracted curve valuation".

If this is not the case, then $f_*\nu$ is finite and we can define: $f_\bullet \nu = \frac{f_*\nu}{c(f,\nu)} \in \mathcal{V}$, where $c(f,\nu) = \nu(f^*\mathfrak{m}) \in [1,+\infty)$

is called the attraction rate of f along ν.

(In fact, $c(f,\nu_{\mathfrak{m}_0}) = c(f)$ the attraction rate)

We extend $f_\bullet$ by continuity to a continuous action $f_\bullet : \mathcal{V} \circlearrowleft$.

This action preserves the type of valuations (but for contracted curve valuations whose image is divisorial)

Geometrically, the action on divisorial valuations can be interpreted as follows: let $\nu_E \in \mathcal{V}^{\mathrm{div}}$: $\exists \pi$ s.t. $E \in \Gamma_\pi^*$.

$$\begin{tikzcd} E \subset X_\pi \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi"] \arrow[dr, "f\circ\pi"] & X_{\pi'} \arrow[d, "\pi'"] \\ (\mathbb{C}^2,0) \arrow[r, "f"] & (\mathbb{C}^2,0) \end{tikzcd}$$

We blow-up the image of E (through $f\circ\pi$ and its lifts) until we get π' so that $f(E) = E' \subset \Gamma_{\pi'}^*$.

Then $f_\bullet \nu_E = \nu_{E'}$

For curves (not contracted), we have $f_\bullet \nu_C = \nu_{f(C)}$.

To prove the theorems we also need to interpret $\mathrm{Ind}(f_\pi)$.

Prop: $f : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, π, π' modifications. Then the lift $\hat{f} : X_\pi \dashrightarrow X_{\pi'}$ is holomorphic at a point $p \in \pi^{-1}(0)$ if and only if $\exists q \in \pi'^{-1}(0)$ s.t. $f_\bullet(U_\pi(p)) \subseteq U_{\pi'}(q)$. In this case, $\hat{f}(p) = q$

$$\begin{tikzcd} X_\pi \arrow[r, dashed, "\hat{f}"] \arrow[d, "\pi"] & X_{\pi'} \arrow[d, "\pi'"] \\ (\mathbb{C}^2,0) \arrow[r, "f"] & (\mathbb{C}^2,0) \end{tikzcd}$$

Here $U_\pi(p)$ is the weak open set obtained as $\overline{U_\pi(p)} \setminus \bigcup_{\substack{E \ni p \\ E \in \Gamma_\pi^*}} \{\nu_E\}$, where

$$\overline{U_\pi(p)} = \{ \nu_E \in \mathcal{V}^{div} \mid \exists\, \pi' = \pi\circ\eta,\ \eta(E) = p \}$$ ← weak open closure

Rem: $\mathcal{V}\setminus\{\nu_E \mid E\in\Gamma_\pi^*\} = \bigcup_{q\in\pi^{-1}(0)} U_\pi(q)$

Proof: (⇒) If $\hat f(p)=q$, then ~~:~~

$\forall \nu\in U_\pi(p)$ can be written as $\nu = \pi_*\mu$, with μ a valuation centered at p

$$f_*\nu = f_*\pi_*\mu = \pi'_*\underbrace{\hat f_*\mu}_{\mu'} = \pi'_*\mu', \quad \mu' \text{ centered at } q.$$

(⇐) Assume $p\in\pi^{-1}(0)$ is an indeterminacy point for $\hat f$. In this case, we can resolve

$$\begin{tikzcd} X_{\tilde\pi} \arrow[d, "\eta"'] \arrow[dr, "\tilde f"] & \\ p\in X_\pi \arrow[r, dashed, "\hat f"] \arrow[d, "\pi"'] & X_{\pi'} \arrow[d, "\pi'"] \\ (\mathbb{C}^2,0) \arrow[r] & (\mathbb{C}^2,0) \end{tikzcd}$$

Blow-up X_π to get a regular lift $\tilde f$. ~~[illegible]~~

Since p was of indeterminacy for $\hat f$, $\hat f(p) = E'\in\Gamma^*_{\pi'}$.

It follows that there exists $E\subset \tilde f^{-1}(E')$ irreducible curve so that $\eta(E)=p$. Then $\nu_E = \pi_*\eta_*\operatorname{ord}_E$ satisfies $f_*\nu_E = \nu_{E'}$, and $\nexists\, q\in\pi'^{-1}(0)$ so that $f_*U_\pi(p)\subseteq U_{\pi'}(q)$. □

Theorem (G-R). Let $f:(\mathbb{C}^2,0)\circlearrowleft$ be a non-invertible germ. Assume f is not finite. Then $\exists!\ \nu_\star = f_\bullet\nu_\star$ such that $f_\bullet^n\nu\to\nu_\star$ $\forall\nu$ quasimonomial

In the ~~[illegible]~~ finite case: $\exists\, I\subset\mathcal{V}$, I either a point or an interval with divisorial or curve ends, s.t. $f^{\circ 2}|_I = \mathrm{id}_I$ and $\forall \nu\in\mathcal{V}^{qm}$, $f_\bullet^n\nu\to I$

(more precisely, $f_\bullet^{2n}\nu \to \overset{\nu_\star}{r_I\nu}$, r_I the retraction to I).

Rem: $\nu_\star$ is not q.m. in general ~~[illegible]~~

The convergence is in the weak topology, and also in a stronger topology whenever $\nu_\star$ is quasimonomial.

Def: a valuation ~~[illegible]~~ $\nu_\star$ as in the theorem is called ~~[illegible]~~ eigenvaluation.

Rem: (Favre-Jonsson): $f_*\nu_\star = c_\infty(f)\cdot\nu_\star$

← first dynamical degree.

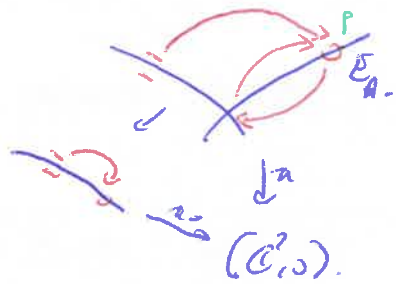
Idea proof of algebraic stability.

Suppose that the theorem gives $\nu_\star = \nu_{E_\star}$ divisorial eigenvaluation.

First, we consider ~~the blow-up~~ a modification $\pi : X_\pi \to (\mathbb{C}^2,0)$ so that $E_\star \in \Gamma_\pi^*$.

In the example: $f(x,y) = (y - x^3; x^2 y)$:

$\nu_\star = \nu_{1,2}$, $E_\star$ is obtained after two blow-ups.



Obstructions to algebraic stability are given by

Indeterminacy points for f_π that are periodic for $f_\pi|_{E_\star}$.

We further blow-up along those orbits, and after sufficiently many blow-ups, we erase this indeterminated periodic orbit.

In this example, it suffices to blow up once:

Notice that at every blow-up, other indeterminacy points may occur.

(□)

Valuative computations:

$$f_* \nu_{1,t} = \begin{cases} t\,\nu_{1,1+\frac{2}{t}} & 1 \le t \le 3 \\ 3\,\nu_{1,\frac{2+t}{3}} & t \geq 3 \end{cases}$$

In particular $f_\bullet U_\pi(p) \ni \nu_{E_\star}$ and $p \in \mathrm{Ind}(f_\pi)$

The action on $E_\star$ corresponds to checking the image of valuations ν_{C_θ}, with $C_\theta = \{y - \theta x^2 = 0\}$. We get that $f_\pi|_{E_\star}(\theta) = \frac{1}{\theta}$.

Rem: When I is a segment, we need π so that $\exists\, \nu_\star = \nu_{E_\star} \in I$, $E_\star \in \Gamma_\pi^*$.

If $f^2|_I = \mathrm{id}$ but $f|_I \neq \mathrm{id}$, we also need for π to satisfy $\nu_{E_\star'} = f_\bullet \nu_{E_\star}$, $E_\star' \in \Gamma_\pi^*$.

This needs further blow-ups, and we still need Γ_π^* to contain the divisors associated to all orbits of new points. This is not possible in general. The model is obtained by quotienting a suitable chain of rational curves.

✳ (end)

Idea of the proof of the valuative theorem.

- Parametrisations: $\forall \nu \in \mathcal{V}$

  - skewness Define $\alpha(\nu) = \sup_{\phi \in \mathfrak{m}} \frac{\nu(\phi)}{\mathfrak{m}(\phi)} = \frac{\nu(\phi)}{\mathrm{ord}_0(\phi)} \in [1; +\infty]$.

  - relative skewness Define $\beta(\nu|\mu) := \sup_{\phi \in \mathfrak{m}} \frac{\nu(\phi)}{\mu(\phi)} \in [1; +\infty]$.

it should be taken on $\mathfrak{m}$-primary ideals to avoid indeterminations, or just take the sup on all $\phi \in \mathfrak{m}$ outside some discrete sets (generic) imprecise

Angular distance: $\rho(\nu, \mu) := \log \beta(\nu|\mu)\beta(\mu|\nu) \in [0; +\infty]$.

Prop (G-R): ρ defines an extended distance on $\mathcal{V}$.

a distance on $\mathcal{V}^\alpha = \{\nu \mid \alpha(\nu) < +\infty\}$.
$\cup$
$\mathcal{V}^{qm}$.

Prop (G-R): let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a germ. Then,

- $\rho(f_\bullet \nu, f_\bullet \mu) \leq \rho(\nu, \mu) \quad \forall \nu, \mu \in \mathcal{V}$.

- If f is non-finite, then the strict inequality holds $\forall \nu \neq \mu \in \mathcal{V}^\alpha$.

Proof of $\leq$: First, remark that $\beta(\nu|\mu)$ is $(1,-1)$-homogeneous. In particular

$$\beta(f_\bullet \nu | f_\bullet \mu) = \frac{c(f, \mu)}{c(f, \nu)} \beta(f_* \nu | f_* \mu).$$

Hence it suffices to show $\beta(f_* \nu | f_* \mu) \cdot \beta(f_* \mu | f_* \nu) \leq \beta(\nu|\mu)\beta(\mu|\nu)$.

$$\beta(f_* \nu | f_* \mu) = \sup_{\phi \in \mathfrak{m}} \frac{f_* \nu(\phi)}{f_* \mu(\phi)} = \sup_{\phi \in \mathfrak{m}} \frac{\nu(\phi \circ f)}{\mu(\phi \circ f)} \leq \sup_{\substack{\psi \in \mathfrak{m} \\ \phi \circ f \in f^* \mathfrak{m} \subseteq \mathfrak{m}}} \frac{\nu(\psi)}{\mu(\psi)} = \beta(\nu|\mu).$$

□

The strict inequality (and in fact a precise characterisation of when we have it for given ν, μ) uses intersection theory of b-divisors.

Prop: (Hodge index/Main lemma) $\nu, \mu_1, \mu_2 \in \mathcal{V}$. Then: $(Z(\nu)\cdot Z(\mu_1))\cdot(Z(\nu)\cdot Z(\mu_2)) \leq (Z(\nu)\cdot Z(\nu))\,(Z(\mu_1)\cdot Z(\mu_2))$,

with equality $\iff$ $\nu = \mu_1$, $\nu = \mu_2$, or μ_1 and μ_2 belong to different connected components of $\mathcal{V} \setminus \{\nu\}$.

- Interpretation of β: $\beta(\nu|\mu) = \dfrac{Z(\nu)\cdot Z(\nu)}{Z(\nu)\cdot Z(\mu)}$.

- Pull back formulas for b-divisors.

Rem: This presentation works also for (X, x_0) singular.

In the smooth case: $\beta(\nu|\mu) = \dfrac{\alpha(\nu)}{\alpha(\nu \wedge \mu)}$ $\Rightarrow$ $\rho = d_{\log\alpha}$, the tree distance associated to the parametrisation $\log\alpha$. ($\wedge$: min.)

Idea proof thm: case of non-finite:

$\forall \pi$ modification, consider $r_\pi : \mathcal{V}_\pi \to S_\pi$ the retraction to the skeleton associated to π ($S_\pi = \{$monomial valuations at p, $p \in \pi^{-1}(0)$, w.r.t. coordinates adapted to $\pi^{-1}(0)\}$)

- $F_\pi = r_\pi \circ f_\bullet : S_\pi \circlearrowleft$ is a weak contraction on a compact space (compact $\subset \mathcal{V}^{qm}$)

$\Rightarrow \exists!\, \nu_\pi = F_\pi \nu_\pi$ fixed, and contracting: $F_\pi^n \nu \to \nu_\pi$ $\forall \nu \in S_\pi$.

- If $\pi' \geq \pi$, ($\pi' = \pi \circ \eta$), then $r_\pi \nu_{\pi'} = \nu_\pi$

- $\nu_\star = \lim_\pi \nu_\pi$ (in the sense of proj. limits for example), satisfies $f_\bullet \nu_\star = \nu_\star$, and local contraction properties.

To get global contraction properties:

The basin of attraction to $\nu_\star$ in $\mathcal{V}^\alpha$ is both open and closed in $\mathcal{V}^2$.

closed: use the equicontinuity.

open: difficult case $\nu_\star$ divisorial.

In this case use log-discrepancy (many) A:

$= A(\mathrm{ord}_E) = 1 + \mathrm{ord}_E(dt \wedge dx) \quad (= 1 + \mathrm{ord}_E(K_{X_\pi/\mathbb{C}^2}))$

$A(\lambda\nu) = \lambda A(\nu)$.

extend by upper semicontinuity to $A : \mathcal{V} \to [2, +\infty]$

We have: Jacobian formula: $A(f_\star\nu) = A(\nu) + \nu(J_f)$

$$\underset{c(f,\nu)\cdot A(f_\bullet\nu)}{\shortparallel} \qquad \underset{\det df}{\shortparallel}$$

$c(f)$ and $\nu(J_f)$ are locally constant outside a finite tree, and

then $A(f_\bullet\nu) - A(f_\bullet\nu_\star) = \dfrac{A(\nu) - A(\nu_\star)}{\underset{\sim 1}{c}}$ a contraction.

$\ast$ Idea of proof of rigidification:

look for invariant $U(p)$ & $f_\bullet : U(p) \circlearrowleft$ is conjugated to $(f_\pi)_\bullet : \mathcal{V}_{\mathfrak{X},p} \circlearrowleft$.

Critical curves for $(f_\pi)_p$ are either the divisorial curves containing p ($\Leftrightarrow \partial U(p)$)

or curves $C_\pi{}^{\to p}$ where C is a critical curve for f.

$\rightsquigarrow$ Need to control $\underset{\underset{\mathcal{V}_f}{\cap}}{\{\nu_C\}}_{C \text{ critical}}$ and $f_\bullet^{-1}(\{\nu_C\})_{C \text{ critical}}$: they are finitely many

local contraction towards $\nu_\star$ $\rightsquigarrow$ Can pick $U_\pi(p)$ so that $\underset{\leq 2}{\#\left[\overline{U_\pi(p)} \cap \left(\mathcal{V}_f \cup f_\bullet^{-1}(\mathcal{V}_f) \cup \partial U_\pi(p)\right)\right]}$

$(f_\pi)_p$ is a

$\rightsquigarrow$ rigid germ.